# NEARLY INTEGRABLE *SO*(3) STRUCTURES ON 5-DIMENSIONAL LIE GROUPS

ANNA FINO AND SIMON G. CHIOSSI

Abstract. Recent work [5] on 5-dimensional Riemannian manifolds with an $SO(3)$ structure prompts us to investigate which Lie groups admit such a geometry. The case in which the $SO(3)$ structure admits a compatible connection with torsion is considered. This leads to a classification under special behaviour of the connection, which enables to recover all known examples, plus others bearing torsion of pure type. Suggestive relations with special structures in other dimensions are highlighted, with attention to eight-dimensional $SU(3)$ geometry.

## 1. Introduction

Given an oriented Riemannian manifold of dimension five $(M^5, g)$, an $SO(3)$ structure is the reduction of the structure group of the frame bundle to the Lie group $SO(3)$ sitting inside $SO(5)$. The inclusion we choose is the one based on the *irreducible* 5-dimensional representation of $SO(3)$, determined by the decomposition

$$\mathfrak{so}(5) = \mathfrak{so}(3) \oplus V,$$

where $V$ is the unique irreducible 7-dimensional representation of $SO(3)$. It is known that the homogeneous space $SO(5)/\,SO(3)$ has an $SO(3)$-connection $\tilde{\nabla}$ whose torsion tensor $T$ is skew-symmetric (and unique), by which one intends that

$$T(X, Y, Z) = g(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y], Z) \tag{1.1}$$

is a three-form for any $X, Y, Z$ vector fields, for which refer to [2, 9]. Indeed this was already noticed in [15], to the effect that the skew-symmetry of $T$ can be used to align $SO(3)$ structures on five-manifolds with other kinds of torsion geometries. In the general framework of geometric structures on Riemannian $n$-manifolds, $G$-reductions are distinguished by the irreducible components of the representation $\mathbb{R}^n \otimes \mathfrak{so}(n)/\mathfrak{g}$, $\mathfrak{g}$ being the Lie algebra of $G$. The decomposition of this space depends on a tensorial object, most of the times a differential form. Well known is the archetypal description of almost Hermitian structures [11] in terms of the Kähler form. Other $G$-geometries have been discussed using the same approach, see [6] amongst others for $G = SU(n)$.

We review $SO(3)$ geometry in section 2. In some sense it is slightly different from more familiar $G$-structures, for it is defined not by means of a skew form - but rather a symmetric tensor, denoted $\mathbb{T}$. Apart from $\mathbb{T}$, one can not expect to find other invariants for the representations of $SO(3)$ on $\mathbb{R}^5$, in particular no differential form. A rather diverse situation occurs in higher dimensions, such as eight for instance, where both totally symmetric and totally skew 3-forms play a relevant role [14], see section 9.

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In the special case of concern, the so-called *nearly integrable* structures, this tensor behaves - strikingly - like the almost complex structure $J$ of a nearly Kähler manifold, whence the similar name.

For a general $SO(3)$-reduction the tensor product $\mathbb{R}^5 \otimes \mathfrak{so}(3)^\perp$ decomposes into the analogues of the Gray–Hervella classes, minding that here the complement $\mathfrak{so}(3)^\perp$ is reducible (hence not $V$). As the torsion $T$ of a nearly integrable $SO(3)$ structure is uniquely defined and skew-symmetric, and the spaces of three- and two-forms on $M^5$ are Hodge-isomorphic, one decomposes the latter under $SO(3)$ in the sum of two irreducible modules

$$\Lambda^2 M^5 = \Lambda^2_3 \oplus \Lambda^2_7$$

of dimensions three, seven respectively. When the components therein are trivial, the characteristic connection $\tilde{\nabla}$ is in fact the Riemannian one, and [5] proves that the simply-connected manifolds $M^5$ admitting a torsion-free $SO(3)$ structure are $\mathbb{R}^5$, $SU(3)/\,SO(3)$ or $SL(3,\mathbb{R})/\,SO(3)$. It is not surprising that these models are symmetric spaces, as proven by Berger's holonomy theorem. The paper of Bobieński and Nurowski classifies 5-dimensional Lie groups with type $\Lambda^2_3$, and finds examples with closed torsion of the complementary type $\Lambda^2_7$.

The present note pertains to 5-dimensional real connected Lie groups $L$ with invariant metric $g$ and $SO(3)$ structure having anti-symmetric torsion $T$. More precisely, we characterise the Lie groups $(L, \mathbb{T}, g)$ admitting a splitting $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{p}$ of the Lie algebra of $L$ defined via an adapted frame, see section 5, in terms of the structure constants. This in turn yields a classification (theorem 4.3) assuming that the $SO(3)$ connection $\tilde{\nabla}$ satisfies

$$\begin{array}{lll} \tilde{\nabla}_X \mathfrak{h} \subseteq \mathfrak{h}, & \tilde{\nabla}_X \mathfrak{p} \subseteq \mathfrak{p}, & \forall X \in \mathfrak{h} \\ \tilde{\nabla}_Y \mathfrak{h} \subseteq \mathfrak{p}, & \tilde{\nabla}_Y \mathfrak{p} \subseteq \mathfrak{h}, & \forall Y \in \mathfrak{p}. \end{array}$$

This special behaviour is displayed by the Levi–Civita connection $\nabla$ in all instances of [5], and corresponds to demanding that the group $L$ act transitively on a Riemannian symmetric surface. We then prove that either the Levi-Civita connection fulfills the same algebraic conditions as $\tilde{\nabla}$, or $\tilde{\nabla}$ is identically zero (theorem 7.1). In the latter case the Lie group is essentially $SO(3) \times \mathbb{R}^2$, and thus the unique instance up to isomorphisms. Our results are in line with the main reference [5] and in some sense attempt to complete that classification. All examples have $d{*}T = 0$, and it would be thus natural to ask whether this were always the case. We prove that a large class of Lie groups, comprising those of [5], satisfies the equivalent requirement of symmetry of the $SO(3)$ connection's Ricci tensor. In every case the torsion type, whether $\Lambda^2_3, \Lambda^2_7$ or generic, is determined. In section 8 we prove the existence of non-strong structures of type $\Lambda^2_7$, by finding an explicit example. This is once again realised by a Lie group that acts transitively on a 3-dimensional symmetric space by way of

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{p}.$$

Section 9 is devoted to understanding the geodesic equation $\nabla_X X = 0$, for any $X$ in $\mathfrak{l}$, which arises from 'near integrability' and has remarkable consequences. Using this property it is possible to construct nearly integrable geometry of higher dimension, namely $SU(3)$ structures on the Riemannian products $L \times \mathbb{R}^3$ and $L \times SO(3)$.

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## 2. Irreducible $SO(3)$ geometry

The vector space $\mathbb{R}^5$ is isomorphic to the set of real $3\times 3$ symmetric matrices with no trace $\mathcal{S}^2_0\mathbb{R}^3$; we fix the isomorphism as follows

$$X=(x_1,\dots,x_5)\longleftrightarrow\begin{pmatrix}\dfrac{x_1}{\sqrt{3}}-x_4 & x_2 & x_3\\ x_2 & \dfrac{x_1}{\sqrt{3}}+x_4 & x_5\\ x_3 & x_5 & -\dfrac{2}{\sqrt{3}}x_1\end{pmatrix}, \tag{2.1}$$

the square root only being a convenient factor. The irreducible representation on $\mathbb{R}^5$ is given by

$$\rho(h)X=hXh^{-1},\quad h\in SO(3),$$

where the vector $X$ is thought of as matrix of the above form. An $SO(3)$ structure on $(M,g)$ can be identified [5] with an element of $\mathbb{T}\in\bigotimes^3\mathbb{R}^5$ such that

i) $\mathbb{T}=\sum_{i,j,k=1}^5 t_{ijk}\,dx_i\otimes dx_j\otimes dx_k$ is symmetric in all arguments,

ii) the endomorphisms $X\mapsto\mathbb{T}_X=X\lrcorner\,\mathbb{T}$ are trace-free and

iii) $(\mathbb{T}_X)^2X=g(X,X)X$, for all $X\in\mathbb{R}^5$.

The contraction $X\lrcorner\,\mathbb{T}$ prescribes to fill the first argument of $\mathbb{T}$, so $\mathbb{T}_X(\cdot,\cdot)=\mathbb{T}(X,\cdot,\cdot)$. Notice by the way that ii) is expected, given that $\sum_j t_{ijj}$ defines the components of an $SO(3)$-invariant vector in $\mathbb{R}^5$, so it must vanish for each $i=1\dots5$ because the representation is irreducible.

These properties together with the identification (2.1) determine an adapted frame, i.e. an orthomormal basis $\{e_1,e_2,e_3,e_4,e_5\}$ of tangent vectors so that $X=\sum_{j=1}^5 x_je_j$ and the metric automatically assumes the canonical form $g=\sum_{i=1}^5(e^i)^2$. Adopting the choice made in [5], the $SO(3)$ structure is completely described by the tensor

$$\mathbb{T}(X,X,X)=x_1(3x_2^2+3x_4^2-x_1^2-\tfrac{3}{2}x_3^2-\tfrac{3}{2}x_5^2)+\tfrac{3\sqrt{3}}{2}x_4(x_5^2-x_3^2)+3\sqrt{3}x_2x_3x_5, \tag{2.2}$$

given by the determinant of $X$.

The action of $SO(3)$ on $\mathbb{C}^2$, like the one of $SU(2)$, endows the symmetric tensor product $\mathcal{S}^4\mathbb{C}^2\cong\mathbb{C}^5$ with a real structure, and $\mathcal{S}^3(\mathcal{S}^4\mathbb{C}^2)$ contains only one copy of $\mathbb{C}$, generated in fact by $\mathbb{T}$. The recipe to tackle a general $G$-structure prescribes first to determine the form(s) with isotropy $G=SO(3)$, the lack of which makes this whole matter quite complicated.

**Definition 2.1.** [5] An $SO(3)$ structure is said *nearly integrable* (*NI*) when

$$(\nabla_X \mathbb{T})(X,X,X) = 0 \tag{2.3}$$

for all vector fields $X \in \mathbb{R}^5$.

This relation is – at least formally – similar to that defining a nearly Kähler structure on an even dimensional manifold: $(\nabla_X J)X = 0$. Just as the almost complex structure $J$ is a Killing form [16] there, (2.3) is saying that $\mathbb{T}$ is a symmetric Killing 3-tensor.

The theoretical interest of such a structure lies in the fact that it admits a uniquely defined *characteristic* connection

$$\tilde{\nabla} = \nabla - \tfrac{1}{2}T \tag{2.4}$$

with torsion $T$ a three-form, in the sense of (1.1). We finally ought to remind that the existence of a nearly integrable $SO(3)$ structure is actually the same [5] as having *skew-symmetric torsion* $T$. Among the simplest examples of $G$-invariant metric connections with anti-symmetric torsion one counts those of naturally reductive spaces. These are homogeneous spaces with a reductive decomposition $\mathfrak{h} \oplus \mathfrak{p}$ possessing a connection $\tilde{\nabla} = \nabla + [\cdot,\cdot]_{\mathfrak{p}}$ and skew-symmetric characteristic torsion tensor $-g([\cdot,\cdot]_{\mathfrak{p}},\cdot)$ by very definition. Because the latter and the induced curvature are parallel, naturally reductive spaces generalise Riemannian symmetric spaces. For a $G$-structure though, there is no Lie group acting transitively. Besides, there are sound reasons to believe that Riemannian manifolds $M^n$ endowed with metric connections $\tilde{\nabla}$ with skew torsion are useful in string and supergravity theories, see [10, 1] and references.

The space $\Lambda^3\mathbb{R}^5$ is isomorphic via the Hodge operator to that of two-forms, and the latter decomposes under $SO(3)$ into the direct sum of the irreducible modules

$$\Lambda^2_3 = \mathrm{span}\{E_1,E_2,E_3\} \cong \mathfrak{so}(3), \qquad \Lambda^2_7 = \left(\Lambda^2_3\right)^{\perp}. \tag{2.5}$$

The orthogonal complement is taken with respect to the pairing

$$\langle \alpha,\beta\rangle = *(\hat{\mathbb{T}}(\alpha)\wedge *\beta), \tag{2.6}$$

via the endomorphism $\hat{\mathbb{T}}(e^i \wedge e^k) = 4\sum_{j,l,m} t_{ijm}t_{klm}e^j \wedge e^l$ of $\Lambda^2(\mathbb{R}^5)^*$ from [5]. Wedge products of basis vectors will be henceforth expressed by juxtaposed indexes.

The forms $E_i$ are given in matrix form by

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\sqrt{3} & 0 & 0 & -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\sqrt{3} & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

or

$$E_1 = \sqrt{3}e^{15} + e^{23} + e^{45}, \quad E_2 = \sqrt{3}e^{13} + e^{25} + e^{34}, \quad E_3 = 2e^{24} + e^{35} \tag{2.7}$$

when seen as elements of $\Lambda^2\mathbb{R}^5$.

We shall say that *a NI structure* $\mathbb{T}$*, the corresponding torsion* $T$ *or the manifold carrying* $\mathbb{T}$ *has/is of type* $\mathcal{W}$ if $*T$ belongs to the irreducible module $\mathcal{W} \subseteq \Lambda^2$. The type can thus be $\Lambda^2_3 \oplus \Lambda^2_7$ generically, $\Lambda^2_3$, $\Lambda^2_7$ which we shall both refer to as 'pure' type, or $\{0\}$ i.e. the torsion-free case.

## 3. The characteristic connection of a 5-dimensional Lie group

We begin by taking a generic Lie group of dimension five and write its structure equations

$$
\begin{cases}
de^1 &= \ b_1e^{12} \ + \ \dots \ + \ b_{10}e^{45} \\
de^2 &= \ b_{11}e^{12} \ + \ \dots \ + \ b_{20}e^{45} \\
de^3 &= \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\
de^4 &= \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\
de^5 &= \ b_{41}e^{12} \ + \ \dots \ + \ b_{50}e^{45}.
\end{cases}
\tag{3.1}
$$

Taking $(e_1, \dots e_5)$ to be the adapted frame means fixing the $SO(3)$ structure and varying the Lie algebra equations in terms of real numbers $b_\alpha, \ \alpha = 1, \dots, 50$.

Polarising the expression (2.2) gives the components of $\mathbb{T}$

$$
\begin{gathered}
t_{111} = -1, \quad t_{122} = 1, \quad t_{144} = 1, \quad t_{133} = -\tfrac{1}{2}, \quad t_{155} = -\tfrac{1}{2}, \\
t_{433} = -\tfrac{\sqrt{3}}{2}, \quad t_{455} = \tfrac{\sqrt{3}}{2}, \quad t_{235} = \tfrac{\sqrt{3}}{2}.
\end{gathered}
$$

**Lemma 3.1.** *The $SO(3)$-tensor $\mathbb{T}$ is NI if and only if the structure equations satisfy the set of linear relations* (3.2) *below.*

*Proof.* Allow $X$ to be a generic linear combination of the basis $(e_i)$ of the Lie algebra $\mathfrak{l}$ of $L$, i.e. $X = \sum_1^5 \lambda_j e_j$. Imposing (2.3) means solving the equation $(\nabla_X \mathbb{T})(X, X, X) = 0$ for all values of the $\lambda_j$'s.

Consider euristically $X = \lambda e_3 + \mu e_4$ and let $Y = \nabla_X X$. Then (2.3) becomes

$$
\begin{aligned}
0 \ &= \tfrac{1}{3}(\nabla_X \mathbb{T})(X, X, X) = \mathbb{T}(Y, X, X) = \lambda^2 \mathbb{T}(Y, e_3, e_3) + 2\lambda\mu \, \mathbb{T}(Y, e_3, e_4) + \mu^2 \mathbb{T}(Y, e_4, e_4) \\
&= \lambda^2(-\tfrac{1}{2} g(Y, e_1) - \tfrac{\sqrt{3}}{2} g(Y, e_4)) + 2\lambda\mu \tfrac{\sqrt{3}}{2} g(Y, e_3) + \mu^2 g(Y, e_1) \\
&= \lambda^2 \big(\tfrac{1}{2} g([X, e_1], X) + \tfrac{\sqrt{3}}{2} g([X, e_4], X)\big) - \sqrt{3}\lambda\mu \, g([X, e_3], X) - \mu^2 g([X, e_1], X),
\end{aligned}
$$

eventually resulting in a fourth order polynomial $\sum_{i+j=4} \lambda^i \mu^j P_{ij}$. Given that $\lambda, \mu$ are arbitrary, the conditions translate into $P_{ij} = 0$ for all $i, j = 0, \dots, 4$. Explicitly

$$
\begin{gathered}
P_{40} = \tfrac{1}{2}(b_{22} - \sqrt{3} b_{28}) = 0, \quad P_{04} = -b_{33} = 0, \quad P_{22} = \tfrac{1}{2} b_{33} - b_{22} + \sqrt{3} b_{28} = 0 \\
P_{31} = \tfrac{1}{2}(b_{32} + b_{23} - \sqrt{3} b_{38}) = 0, \quad P_{13} = -(b_{23} + b_{32}) + \sqrt{3} b_{38} = 0.
\end{gathered}
$$

The story is similar when all $\lambda_i$'s are present. Passing from linear combinations $X$ of $p$ elements to $p+1$ does not increase dramatically the complexity, since most of the new information is trivial by previous relations, justifying the fact that the system is linear. The

set of conditions imposed by (2.3) reads thus:

$$
\begin{gathered}
b_1 = b_{11} = b_3 = b_{33} = 0,\ b_{20} = -b_{37},\ b_{13} + b_{31} = 0,\\
b_2 = \sqrt{3}(b_{23} + b_8),\ b_4 = \sqrt{3}(-b_{43} + b_{10}),\ b_{22} = \sqrt{3}b_{28},\ b_{44} = \sqrt{3}b_{50},\\
b_{21} + b_{12} = \sqrt{3}b_{17},\ b_{14} + b_{41} = \sqrt{3}b_{15},\ b_4 = \sqrt{3}(b_5 - b_{21}),\ b_2 = \sqrt{3}(b_7 - b_{41}),\\
2b_{22} + \sqrt{3}b_{16} = 2\sqrt{3}(b_{19} + b_{27}),\ 2b_{44} - \sqrt{3}b_{16} = 2\sqrt{3}(b_{45} - b_{19}),\\
2b_{29} + b_{17} = b_{26} + b_{18},\ 2b_{29} + b_{40} = b_{26} + b_{35},\ 2b_{49} - b_{15} - b_{37} = b_{46},\\
b_{28} + b_{50} = b_{45} + b_{27},\ b_{24} + b_{42} = \sqrt{3}(b_{25} + b_{47}),\ b_{48} - b_{30} = b_{47} - b_{25},\\
2(b_{24} + b_9) = 2\sqrt{3}b_{25} + b_{13} + b_6,\ 2(b_{42} - b_9) = 2\sqrt{3}b_{47} - (b_{13} + b_6),\\
2(b_{39} + b_{30} - b_{25}) = b_{36},\ b_{35} + b_{17} = b_{40} + b_{18},\ 2(b_{48} + b_{39} - b_{47}) = b_{36},\\
b_{38} = -b_{15},\ \sqrt{3}(b_{40} + b_{18} - b_{35}) = b_{21} + b_{12},\ b_{32} = -b_{23} - \sqrt{3}b_{15},
\end{gathered}
\tag{3.2}
$$

easily handled by computer programs. □

Further constraints on the coefficients in (3.1) derive from $d^2 = 0$, but due to computational complexity we reserve the Jacobi identity to when strictly necessary, typically at the very end of the classifying process.

Since $\tilde\nabla$ preserves the metric and the tensor $\mathbb{T}$, it is clear that

$$
\begin{gathered}
g(\tilde\nabla_X Y, Z) + g(\tilde\nabla_X Z, Y) = 0,\\
\mathbb{T}(\tilde\nabla_X Y, Z, W) + \mathbb{T}(\tilde\nabla_X Z, Y, W) + \mathbb{T}(\tilde\nabla_X W, Y, Z) = 0.
\end{gathered}
$$

The characteristic torsion (2.4) of the structure is then given by

$$
\begin{aligned}
T = \quad & (b_{43} - b_{10} + b_{12})e^{123} - b_6 e^{124} + (\sqrt{3}b_{15} - b_7)e^{125} +\\
& (\sqrt{3}b_{15} - b_8)e^{134} + (b_{24} - \sqrt{3}b_{47} - \tfrac{1}{2}b_{13} - \tfrac{1}{2}b_6)e^{135} +\\
& (\sqrt{3}b_{40} - b_{10})e^{145} + (2b_{29} - b_{17} - b_{35})e^{234} + (b_{28} - b_{50} - b_{19})e^{235} +\\
& (b_{37} - b_{15} - 2b_{49})e^{245} + (\tfrac{\sqrt{3}}{6}b_{13} + \tfrac{\sqrt{3}}{6}b_6 - \tfrac{\sqrt{3}}{3}b_9 - \tfrac{\sqrt{3}}{3}b_{24} + b_{39} - b_{47})e^{345}.
\end{aligned}
$$

The $SO(3)$-connection $\tilde\nabla$ can be thus reconstructed, and is written here as an $\mathfrak{so}(3)$-valued form

$$\Gamma = \gamma^1 E_1 + \gamma^2 E_2 + \gamma^3 E_3$$

where

$$\gamma^1 = \Gamma^2_3, \qquad \gamma^2 = \Gamma^2_5, \qquad \gamma^3 = \Gamma^3_5 \tag{3.3}$$

and

$$
\begin{aligned}
&\Gamma^2_5 = \tfrac{1}{\sqrt{3}}\Gamma^1_3 = \Gamma^3_4 = -(b_{23} + b_8)e^1 - b_{17}e^2 - b_{28}e^3 + b_{15}e^4 - b_{47}e^5,\\
&\Gamma^2_3 = \tfrac{1}{\sqrt{3}}\Gamma^1_5 = \Gamma^4_5 = (b_{43} - b_{10})e^1 - b_{15}e^2 + \tfrac{1}{\sqrt{3}}(-b_9 + \tfrac{1}{2}b_{13} + \tfrac{1}{2}b_6 - b_{24})e^3 - b_{40}e^4 - b_{50}e^5,\\
&\Gamma^3_5 = \tfrac{1}{2}\Gamma^2_4 = -\tfrac{1}{2}(b_6 + b_{13})e^1 + (b_{45} - b_{50} - b_{19})e^2 - b_{29}e^3 + (b_{47} - b_{39} - b_{48})e^4 - b_{49}e^5
\end{aligned}
$$

are the connection 1-forms. All $\Gamma^i_j$'s vanish precisely when

$$
\begin{gathered}
b_{43} = b_{10}, b_{23} = -b_8, b_{15} = b_{17} = b_{29} = b_{40} = b_{49} = 0,\\
b_{28} = b_{47} = b_{50} = 0, b_{45} = b_{19}, b_{48} = -b_{39}, b_{13} = -b_6, b_{24} = -b_9.
\end{gathered}
\tag{3.4}
$$

Now the curvature $K = r^1E_1 + r^2E_2 + r^3E_3 \in \Lambda^2\mathfrak{l}^* \otimes \mathfrak{so}(3)$ is determined by

$$r^1 = d\gamma_1 + \gamma_2 \wedge \gamma_3, \quad r^2 = d\gamma_2 + \gamma_3 \wedge \gamma_1, \quad r^3 = d\gamma_3 + \gamma_1 \wedge \gamma_2,$$

and $r^1 = K^2_3, r^2 = K^2_5, r^3 = K^3_5$ are the non-zero components of $K$. If $T \equiv 0$ then $K$ is determined by a constant $F \in \mathbb{R}$, since [5]

$$r^j = FE_j \qquad j = 1, 2, 3$$

and the $SO(3)$ structure is locally isometric to that of a symmetric space $Q/\,SO(3)$, with

$$Q = \begin{cases} SO(3) \times_\rho \mathbb{R}^5 & \text{if } F = 0 \\ SL(3, \mathbb{R}) & \text{if } F < 0 \\ SU(3) & \text{if } F > 0. \end{cases} \tag{3.5}$$

A purely formal but still suggestive relation between near integrability and lower dimensional geometry is the fact that the number of parameters $b_\alpha$ upon which the Riemannian connection of a *NI* structure depends, is precisely half of the total, exactly as for self-dual connections on a four-manifold [14]. Whether this has to do with the universal covering $SU(2) \to SO(3)$ remains to be seen, though the fact that the Berger sphere $SO(5)/\,SO(3)$ is diffeomorphic to $Sp(2)/\,SU(2)$ is a clear indication, and will be dealt with elsewhere by the authors. But here is an

*Example.* Consider the Lie algebra $\mathfrak{l}$ with structure equations

$$\begin{cases} de^1 = b_7e^{25} - b_{37}e^{35} - b_{45}e^{45}, \\ de^2 = -b_7e^{15} + b_{45}e^{35} - b_{37}e^{45}, \\ de^3 = b_{37}e^{15} - b_{45}e^{25} + \frac{(b_{37}^2+b_{45}^2)}{b_7}e^{45}, \\ de^4 = b_{45}e^{15} + b_{37}e^{25} - \frac{(b_{37}^2+b_{45}^2)}{b_7}e^{35}, \\ de^5 = b_7e^{12} - b_{37}e^{24} - b_{45}e^{14} - b_{37}e^{13} + b_{45}e^{23} + \frac{(b_{37}^2+b_{45}^2)}{b_7}e^{34}, \end{cases} \tag{3.6}$$

and the *NI* structure whose associated three-form $T$ is given by

$$T = -b_7e^{125} + b_{37}e^{135} + b_{45}e^{145} - b_{45}e^{235} + b_{37}e^{245} - \frac{(b_{37}^2+b_{45}^2)}{b_7}e^{345}.$$

First of all this torsion is harmonic. But more crucially, the associated Lie group $L$ is thus endowed with an $SU(2)$ structure defined by

$$\alpha = e^5, \quad \omega_1 = e^{12} + e^{34},\ \omega_2 = e^{13} + e^{42},\ \omega_3 = e^{14} + e^{23}.$$

To be precise, this is actually hypo [8] as

$$d\omega_1 = 0, \quad d(\omega_2 \wedge \alpha) = d(\omega_3 \wedge \alpha) = 0.$$

Examples of this sort are noteworthy because, being hypo, they induce local integrable $SU(3)$ structures on $L \times \mathbb{R}$. Moreover, they yield half-flat geometries, which also evolve in one dimension higher, but to holonomy $G_2$. In fact, (3.6) is part of a more general family, as shown by

**Proposition 3.2.** *Let* $L$ *be a Lie group whose Maurer-Cartan structure* (3.1) *satisfies* (3.4) *plus* $b_6 = b_8 = b_{12} = b_{35} = 0, b_{10} = -b_{45}, b_9 = -b_{37}, b_{39} = (b_{37}^2 + b_{45}^2)/b_7$. *Then for some closed* 1*-form* $e^6$*, the invariant* $SU(3)$ *structure on* $L \times \mathbb{R}$

$$\omega = \omega_3 + \alpha \wedge e^6, \qquad \psi_+ = \omega_2 \wedge \alpha - \omega_1 \wedge e^6$$

*is half-flat.*

*Proof.* Almost immediate, once one recalls that half-flat is a fancy name for the closure of $\psi_+$ and $\omega \wedge \omega$. $\square$

## 4. Classification

The action of the endomorphisms $\mathbb{T}_X = \mathbb{T}(X, \cdot, \cdot)$ indicates that whilst $\mathbb{T}_{e_1}, \mathbb{T}_{e_2}$ and $\mathbb{T}_{e_4}$ preserve the decomposition

$$\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{p} = \mathrm{span}\{e_1, e_2, e_4\} \oplus \ \mathrm{span}\{e_3, e_5\}, \tag{4.1}$$

$\mathbb{T}_{e_3}, \mathbb{T}_{e_5}$ on the contrary do not. This reflects the irreducibility of the rotational action.

We shall capture the Lie algebras $\mathfrak{l}$ for which the $SO(3)$-connection satisfies the conditions:

$$\begin{array}{lll} \tilde{\nabla}_X \mathfrak{h} \subseteq \mathfrak{h}, & \tilde{\nabla}_X \mathfrak{p} \subseteq \mathfrak{p}, & \forall X \in \mathfrak{h} \\ \tilde{\nabla}_Y \mathfrak{h} \subseteq \mathfrak{p}, & \tilde{\nabla}_Y \mathfrak{p} \subseteq \mathfrak{h}, & \forall Y \in \mathfrak{p}. \end{array} \tag{4.2}$$

Otherwise said, $\mathfrak{h}$-derivatives preserve the splitting (4.1), but differentiation in the $\mathfrak{p}$-direction swaps the subspaces. The geometric motivation of this condition are readily explained.

In order to classify 5-dimensional Lie groups $L$ with an invariant $SO(3)$ structure $(g, \mathbb{T})$ whose characteristic connection $\tilde{\nabla}$ satisfies (4.2), it is quite useful to distinguish whether $(\mathfrak{l}, \mathfrak{h})$ is a symmetric pair or not. A Lie algebra $\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{q}$, or better $(\mathfrak{l}, \mathfrak{m})$ is called a *symmetric pair* when

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{q}] \subseteq \mathfrak{q}, \quad [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{m}.$$

If $H$ is the simply-connected Lie subgroup of $L$ with Lie algebra $\mathfrak{h}$, we have a Riemannian fibration of Lie groups

$$H \longrightarrow L \longrightarrow L/H \tag{4.3}$$

whose fibres $H$ are totally geodesic submanifolds of $L$ and whose base is a 2-dimensional symmetric space. Therefore $L/H$ is a finite quotient of $\mathbb{R}^2$, the Riemann sphere or the Poincaré disc according to the curvature. Since $\mathfrak{p}$ is indeed the tangent 2-plane spanned by $e_3, e_5$, the curvature of $L/H$ is given by

$$k(\mathfrak{p}) = -g(e_5, \left[[e_3, e_5]_{\mathfrak{h}}, e_3\right]_{\mathfrak{p}}) = b_9(\tfrac{1}{2}b_6 + \tfrac{1}{2}b_{13} - b_9 - \sqrt{3}b_{47}) - b_{19}b_{45} + b_{39}b_{48}, \tag{4.4}$$

and its sign decides which of the space forms one is looking at.

**Lemma 4.1.** *If $\mathfrak{p}$ is $ad(\mathfrak{h})$-invariant then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{l}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$.*

*Proof.* If $\mathfrak{p}$ is $ad(\mathfrak{h})$-invariant, the structure coefficients

$$b_2, b_4, b_5, b_7, b_8, b_{10}, \ b_{12}, b_{14}, b_{15}, b_{17}, b_{18}, \ b_{32}, b_{34}, b_{35}, b_{37}, b_{38}, b_{40}$$

all vanish. As a consequence of (3.2) also $b_{23}, b_{43}, b_{21}, b_{41}$ are zero. So the Maurer–Cartan system (3.1) simplifies to

$$\begin{cases} de^1 = & b_6e^{24} + b_9e^{35} \\ de^2 = & b_{13}e^{14} + 2(b_{50} + b_{19} - b_{45})e^{24} + b_{19}e^{35} \\ de^3 = & \sqrt{3}b_{28}e^{13} + b_{24}e^{15} + \frac{\sqrt{3}}{3}(b_9 + b_{24} - \frac{1}{2}(b_{13} + b_6))e^{23} + 2b_{29}e^{24} + (b_{50} + b_{28} - b_{45})e^{25} + \\ & b_{28}e^{34} + b_{29}e^{35} + \frac{\sqrt{3}}{3}(b_9 + b_{24} - \frac{1}{2}(b_{13} + b_6) + \sqrt{3}b_{48} - \sqrt{3}b_{47})e^{45} \\ de^4 = & -b_{13}e^{12} + 2(b_{39} + b_{48} - b_{47})e^{24} + b_{39}e^{35} \\ de^5 = & (b_9 + \sqrt{3}b_{47} - \frac{1}{2}b_{13} - \frac{1}{2}b_6)e^{13} + \sqrt{3}b_{50}e^{15} + b_{45}e^{23} + 2b_{49}e^{24} + b_{47}e^{25} + b_{48}e^{34} + \\ & b_{49}e^{35} + b_{50}e^{45}. \end{cases}$$

Using the vanishing of the coefficients of $e^{234}, e^{245}$ in $d(de^i)$, $i = 1, 2, 4$, and of $e^{124}, e^{135}, e^{235}, e^{345}$ for $i = 3, 5$, then either

- $b_{29} = b_{49} = 0$ (implying $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$), or
- $b_{29}^2 + b_{49}^2 \neq 0$ and $b_9 = b_{19} = b_{39} = 0, b_{13} = -b_6, b_{45} = -b_{28}, b_{48} = b_{47}, b_{50} = -b_{28}$ (which force $\mathfrak{p}$ to become a Lie subalgebra of $\mathfrak{l}$).

But in the latter case, the Jacobi equation ends up annihilating both $b_{49}$ and $b_{29}$, contradicting the assumption. $\square$

**Corollary 4.2.** *Demanding* $\mathfrak{p}$ *to be* $ad(\mathfrak{h})$*-invariant is equivalent to the pair* $(\mathfrak{l}, \mathfrak{h})$ *being symmetric.*

*Proof.* In terms of the structure equations $\tilde{\nabla}$ satisfies (4.2) if and only if

$$b_{43} = b_{10}, b_{23} = -b_8, b_{15} = b_{17} = b_{29} = b_{37} = b_{40} = b_{49} = 0.$$

If, in addition, $b_7 = b_8 = b_{10} = b_{12} = b_{35} = 0$ then $[\mathfrak{p}, \mathfrak{h}] \subseteq \mathfrak{p}$ and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{l}$. $\square$

The algebraic structures appearing later will often be *solvable* Lie algebras $\mathfrak{g}$, whose derived series

$$\mathfrak{g} \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \ldots \supseteq \mathfrak{g}^q = 0$$

collapses at some stage $q \in \mathbb{N}$ called the step length. Each ideal is defined by bracketing the preceding one in the sequence with itself, $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i]$, beginning from $\mathfrak{g}^0 = \mathfrak{g}$. The first subspace $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is the commutator and such a $\mathfrak{g}$ is referred to as $q$-step (solvable).

*Examples.* There is only one solvable (non-nilpotent) Lie algebra of dimension two, modulo isomorphisms:

$$\mathfrak{s}_2 = (0, e^{12}),$$

and similarly, only one 3-dimensional 2-step one, namely

$$\mathfrak{s}_3 = (0, 0, e^{13}).$$

This notation for Lie algebras expresses the bracket of $\mathfrak{g}$ via the exterior differential of $\mathfrak{g}^*$, so in the latter example one should understand the basis $e_1, e_2, e_3$ of $\mathfrak{s}_3$ to satisfy $de^1 = de^2 = 0, de^3 = e^{13}$, or equivalently $[e_1, e_2] = [e_2, e_3] = 0, [e_1, e_3] = -e_3$.

Solvable Lie algebras and groups are intimately linked to non-positive curvature, as predicted by Alekseevskiĭ [3]. This phenomenon manifests itself patently in the coming sections, goal of which is to prove the

**Theorem 4.3.** *If the characteristic connection behaves as in* (4.2)*, the torsion is always coclosed* $d{*}T = 0$*. Moreover*
*a) if* $(\mathfrak{l}, \mathfrak{h})$ *is symmetric,* $\mathfrak{l}$ *is either solvable or isomorphic to*

$$\begin{aligned}
\mathfrak{l}_1 &= \mathfrak{so}(3) \oplus \mathbb{R}^2\\
\mathfrak{l}_2 &= (2e^{12}, e^{14}, e^{15} - e^{23}, 2e^{24}, e^{25} + e^{34}),\\
\mathfrak{l}_3 &= \mathfrak{so}(3) \oplus \mathfrak{s}_2
\end{aligned}$$

*b) If* $(\mathfrak{l}, \mathfrak{h})$ *is not symmetric instead, then* $\tilde{\nabla} \equiv 0$ *and* $\mathfrak{l}$ *is isomorphic to* $\mathfrak{so}(3) \oplus \mathbb{R}^2$.

The appearance of algebras isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$ in distinct contexts depends upon the underlying $SO(3)$ structure, which changes in *a)* and *b)*. The proof of this statement is scattered over sections 5 and 7.

The type of $SO(3)$ geometry is determined by the presence of intrinsic torsion in the modules (2.5). Because the two-forms $E_i$'s are eigenvectors of the endomorphism $\hat{\mathbb{T}}$ of (2.6) with a known eigenvalue, one has that ${*}T \in \Lambda^2_7$ if and only if

$$T \wedge E_i = 0, \; i = 1, 2, 3. \tag{4.5}$$

The Lie algebras of type $\Lambda^2_7$ are therefore those for which this linear system in the $b_\alpha$'s holds. They will be highlighted in the rest of the paper.

To conclude the section we recall two basic equations [13] for metric connections with torsion $\tilde{\nabla}$. One relates the covariant and exterior derivatives of $T$

$$dT(x, y, z, w) = \mathfrak{S}_{x,y,z}(\tilde{\nabla}_x T)(y, z, w) - (\tilde{\nabla}_w T)(x, y, z) + 2\sigma_T(x, y, z, w), \tag{4.6}$$

where $\mathfrak{S}$ is the cyclic sum. The four-form

$$2\sigma_T = \textstyle\sum_1^5 (e_i \lrcorner T)^2 \tag{4.7}$$

also shows up in a second equation, that is the Bianchi identity for the curvature $\tilde{R}$ of $\tilde{\nabla}$

$$\mathfrak{S}_{x,y,z}\tilde{R}(x, y, z, w) = dT(x, y, z, w) - \sigma_T(x, y, z, w) + (\tilde{\nabla}_w T)(x, y, z).$$

**Corollary 4.4.** [13] *When the torsion is* $\tilde{\nabla}$*-parallel then* $dT = 2\sigma_T$.
*If additionally the manifold happens to be flat* $\tilde{R} = 0$*, then it has closed torsion.* □

For the significance of $\sigma_T$ see for instance [2].

## 5. $(\mathfrak{l}, \mathfrak{h})$ symmetric pair

The case when $(\mathfrak{l}, \mathfrak{h})$ is a symmetric pair corresponds to the Levi-Civita connection satisfying a bunch of relations similar to (4.2):

**Proposition 5.1.** *The Levi-Civita connection* $\nabla$ *satisfies*

$$\begin{array}{lll}
\nabla_X \mathfrak{h} \subseteq \mathfrak{h}, & \nabla_X \mathfrak{p} \subseteq \mathfrak{p}, & \forall X \in \mathfrak{h}\\
\nabla_Y \mathfrak{h} \subseteq \mathfrak{p}, & \nabla_X \mathfrak{p} \subseteq \mathfrak{h}, & \forall Y \in \mathfrak{p}
\end{array}$$

*if and only if* $\mathfrak{p}$ *is* $\mathrm{ad}(\mathfrak{h})$*-invariant and* $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{l}$.

*Proof.* The conditions are equivalent to the vanishing of the coefficients $b_7$, $b_8$, $b_{10}$, $b_{12}$, $b_{15}$, $b_{17}$, $b_{23}$, $b_{29}$, $b_{35}$, $b_{37}$, $b_{40}$, $b_{43}$, $b_{49}$ in (3.1). □

The first relation is an old acquaintance, for $0 = (\nabla_{\mathfrak{h}}\mathfrak{h})_{\mathfrak{p}}$ is the second fundamental form of $H \subset L$.

Imposing (4.2) reduces the characteristic connection to the treatable

$$
\begin{aligned}
\tilde\nabla e_1 &= -\sqrt{3}(b_{28}e_3 + Ae_5)\otimes e^3 + \sqrt{3}(b_{47}e_3 + b_{50}e_5)\otimes e^5\\
\tilde\nabla e_2 &= 2Be_4\otimes e^1 + 2Ce_4\otimes e^2 + (Ae_3 + b_{28}e_5)\otimes e^3 + 2De_4\otimes e^4 +\\
&\qquad (b_{50}e_3 + b_{47}e_5)\otimes e^5,\\
\tilde\nabla e_3 &= Be_5\otimes e^1 + Ce_5\otimes e^2 + \big(b_{28}(e_4 - \sqrt{3}e_1) - Ae_2\big)\otimes e^3 + De_5\otimes e^4 +\\
&\qquad \big(b_{47}(e_4 - \sqrt{3}e_1) - b_{50}e_2\big)\otimes e^5,\\
\tilde\nabla e_4 &= -2Be_2\otimes e^1 - 2Ce_2\otimes e^2 + (-b_{28}e_3 + Ae_5)\otimes e^3 - 2De_2\otimes e^4 +\\
&\qquad (b_{50}e_5 - b_{47}e_3)\otimes e^5,\\
\tilde\nabla e_5 &= -Be_3\otimes e^1 - Ce_3\otimes e^2 - \big(b_{28}e_2 + A(e_4 - \sqrt{3}e_1)\big)\otimes e^3 - De_3\otimes e^4 -\\
&\qquad \big(b_{47}e_2 + b_{50}(e_4 + \sqrt{3}e_1)\big)\otimes e^5,
\end{aligned}
\tag{5.1}
$$

where capitals are merely used to abbreviate the constants

$$B = \tfrac{1}{2}(b_6 + b_{13}),\quad A = \tfrac{\sqrt{3}}{3}(b_9 + b_{24} - B),\quad C = b_{50} - b_{45} + b_{19},\quad D = b_{39} + b_{48} - b_{47}.$$

**Corollary 5.2.** *The splitting $\mathfrak{h}\oplus\mathfrak{p}$ is curvature-invariant:*

$$R_{XY}(\mathfrak{h}) \subseteq \mathfrak{h},\ R_{XY}(\mathfrak{p}) \subseteq \mathfrak{p} \qquad \forall X, Y \in \mathfrak{l}.$$

*Proof.* From (4.2) one induces easily that

$$
\begin{aligned}
&R_{\mathfrak{h}\mathfrak{h}}\mathfrak{h},\ R_{\mathfrak{h}\mathfrak{p}}\mathfrak{h},\ R_{\mathfrak{p}\mathfrak{p}}\mathfrak{h} \in \mathfrak{h},\\
&R_{\mathfrak{h}\mathfrak{p}}\mathfrak{p},\ R_{\mathfrak{h}\mathfrak{h}}\mathfrak{p},\ R_{\mathfrak{p}\mathfrak{p}}\mathfrak{p} \in \mathfrak{p}.
\end{aligned}
$$

Using the symmetries of $R$, this means $R_{\mathfrak{h}\mathfrak{h}\mathfrak{h}\mathfrak{p}} = R_{\mathfrak{h}\mathfrak{p}\mathfrak{p}\mathfrak{p}} = 0$, i.e. whenever any three indexes in the components $R_{ijkl}$ denote vectors in $\mathfrak{p}$ (or $\mathfrak{h}$), the curvature is zero. In other words the curvature 2-form preserves (4.1). □

As a consequence, the $SO(3)$-curvature $\tilde R$ fulfills similar conditions $\tilde R_{\mathfrak{h}\mathfrak{h}\mathfrak{h}\mathfrak{p}} = \tilde R_{\mathfrak{h}\mathfrak{p}\mathfrak{h}\mathfrak{h}} = \tilde R_{\mathfrak{h}\mathfrak{p}\mathfrak{p}\mathfrak{p}} = \tilde R_{\mathfrak{p}\mathfrak{p}\mathfrak{h}\mathfrak{p}} = 0$, despite it no longer being a symmetric endomorphism of $\Lambda^2\mathfrak{l}$.

The vector space decomposition (4.1) transfers also to the algebraic level as follows

**Corollary 5.3.** *When $\mathfrak{p}$ is Abelian $\mathfrak{l}$ is a semidirect sum*

$$\mathfrak{l} = \mathfrak{h}\oplus_\alpha \mathfrak{p},$$

*where $\alpha : \mathfrak{h} \to \mathfrak{Der}(\mathfrak{p})$ is a homomorphism of Lie algebras.*

*Proof.* This descends from the $ad(\mathfrak{h})$-invariancy of $\mathfrak{p}$. □

For better handling the discussion now divides into the mutually exclusive cases

$$T = 0, \qquad dT = 0 \ (T \neq 0, \text{ called ‘strong’}), \quad \text{and} \quad dT \neq 0,$$

the simplest being the torsion-free one.

**Proposition 5.4.** *Let $(\mathfrak{l}, \mathfrak{h})$ be a symmetric pair with $T \equiv 0$. Then*

(i) [5] *the $SO(3)$ structure of $L$ is locally isometric to that of a symmetric space of non-positive curvature, hence either $\mathbb{R}^5$ or $SL(3,\mathbb{R})/SO(3)$;*

(ii) $\mathfrak{l}$ *is solvable;*
(iii) *the symmetric space $L/H$ is flat.*

*Proof.* The structure equations are

$$\begin{cases} de^1 = b_9 e^{35} \\ de^2 = b_{13}e^{14} + 2(b_{50}+b_{19}-b_{45})e^{24} + (b_{28}-b_{50})e^{35} \\ de^3 = \sqrt{3}b_{28}e^{13} + (\frac{1}{2}b_{13}+\sqrt{3}b_{47})e^{15} + (-b_{47}+\frac{\sqrt{3}}{3}b_9)e^{23} + (b_{50}+b_{28}-b_{45})e^{25} + \\ \qquad b_{28}e^{34} + (\frac{\sqrt{3}}{3}b_9 + b_{48})e^{45} \\ de^4 = -b_{13}e^{12} + 2(b_{39}+b_{48}-b_{47})e^{24} + b_{39}e^{35} \\ de^5 = (b_9 + 2\sqrt{3}b_{47} - b_{24})e^{13} + \sqrt{3}b_{50}e^{15} + b_{45}e^{23} + b_{47}e^{25} + b_{48}e^{34} + b_{50}e^{45}. \end{cases}$$

The inspection of the Jacobi identity tells that there are four solutions for which $\nabla = \tilde{\nabla}$. Only the non-zero coefficients are indicated and serve to distinguish the Lie algebras:

(1) $b_{28} = b_{50} = a \neq 0$
$(0, 2ae^{24}, a(\sqrt{3}e^{13} + e^{34} + 2e^{25}), 0, a(\sqrt{3}e^{15} + e^{45}))$

(2) $b_{28} = b_{50} = \frac{1}{2}(b_{45}^2 + b_{48}^2)/b_{45}, b_{45} = a \neq 0, b_{48} = b$
$(0, \frac{-a^2+b^2}{a}e^{24}, be^{45} + \frac{a^2+b^2}{2a}(e^{34} + \sqrt{3}e^{13}) + \frac{b^2}{a}e^{25}, 2be^{24}, \frac{a^2+b^2}{2a}(e^{45} + \sqrt{3}e^{15}) + be^{34} + ae^{23})$.

(3) $b_{50} = -\frac{1}{2}b_{19}, b_{28} = \frac{1}{2}b_{19}, b_{39} = 2b_{47}, b_{24} = \sqrt{3}b_{47}, b_{19} = a, b_{47} = b, a^2 + b^2 \neq 0$
$(0, a(e^{24} + e^{35}), \frac{1}{2}a(\sqrt{3}e^{13} + e^{34}) + b(\sqrt{3}e^{15} + e^{23}), b(2e^{24} + 2e^{35}), b(\sqrt{3}e^{13} + e^{25}) - \frac{1}{2}a(\sqrt{3}e^{15} + e^{45}))$.

(4) $b_{13} = 2b_{24} = a \neq 0$
$(0, ae^{14}, \frac{1}{2}ae^{15}, -ae^{12}, -\frac{1}{2}ae^{13})$.

The first three instances are 3-step solvable Lie algebras with 3-dimensional commutator $\mathfrak{l}^1$ and $\mathfrak{h} \cong \mathbb{R} \oplus \mathfrak{s}_2$. The Lie algebra (1) is isomorphic to (2) with $b = 0$. The Lie algebra (4) is 2-step solvable with 4-dimensional commutator $\mathfrak{l}^1$ and $\mathfrak{h}$ Abelian. Instead $\mathfrak{p}$ is an Abelian subalgebra only for the Lie algebras (1), (2) and (4), so the curvature formula (4.4) implies sectional flatness. In the remaining case (where $\mathfrak{p}$ is not Abelian) we still have $k(\mathfrak{p}) = 0$. The constant $F$ of (3.5) is negative for (1)-(3) and vanishes for (4), so [5] implies that the first three are isometric to $SL(3, \mathbb{C})/SO(3)$, the last to $\mathbb{R}^5$. □

The Lie algebras listed (in roman numerals) in the sequel have simpler structure equations than those generated in the proofs. They are attained by standard changes of bases – omitted not to bore the reader senseless – and may not be optimal: for example $\sqrt{3}e^1 + e^4$ could reasonably be a basic 1-form, as variously hinted by, e.g., (5.1). This depends essentially on the choice of representation (2.1).

Nevertheless, it should be all but clear that doing this will alter the $SO(3)$ structure rather dramatically, so retaining in the proofs the original expressions, prior to any algebra isomorphism, allows to read off the geometry of interest. In order to truly distinguish algebras up to $SO(3)$ equivalence, Cartan-Kähler theory seems the only reasonable way. This was pursued in [5].

**Lemma 5.5.** *If $(\mathfrak{l},\mathfrak{h})$ is a symmetric pair with strong torsion, then $\mathfrak{l}$ must be isomorphic to one of*

I. $(0, -2e^{24}, (e^{13}+e^{34}) + m(e^{15}+e^{23}), -2me^{24}, m(e^{13}+e^{25}) - (e^{15}+e^{45}))$

II. $(e^{24}, e^{14}, 0, e^{12}, 0)$

III. $(e^{35}, 0, e^{45}+e^{23}, e^{24}, -e^{25})$

IV. $(-ce^{35}, ae^{35}, ce^{15}-ae^{25}+be^{45}, -be^{35}, -ce^{13}+ae^{23}+be^{34})$,

*where $m, a, b, c \in \mathbb{R}, a^2+b^2+c^2 \neq 0$. The first two algebras give rise to a flat quotient $L/H$, whereas $k(\mathfrak{p})$ is negative for III and IV. The torsion is of pure type $\Lambda^2_7$ in the family of Lie algebras IV when $c = 0$, and always in I.*

*Proof.* Suppose that $T$ be closed, and not identically zero. After imposing the Jacobi identity one reduces to four instances:

(1) $b_{50} = -b_{28} = -a \neq 0, b_{24} = \sqrt{3}b_{47} = \sqrt{3}b$
$(0, -2ae^{24}, a(\sqrt{3}e^{13}+e^{34}) + b(\sqrt{3}e^{15}+e^{23}), -2be^{24}, b(\sqrt{3}e^{13}+e^{25}) - a(\sqrt{3}e^{15}+e^{45}))$ with $T = 2(ae^2+be^4)e^{35}$ and $k(\mathfrak{p}) = 0$. Example 6.3.3 in [5] is contained in this family of algebras by requiring $b \neq 0$.

(2) $b_6 = -b_{13} = a \neq 0$
$(-ae^{24}, ae^{14}, 0, -ae^{12}, 0)$ with $T = ae^{124}$ and $k(\mathfrak{p}) = 0$.

(3) $b_{47} = -\frac{1}{3}\sqrt{3}b_9 = -a \neq 0$
$(\sqrt{3}ae^{35}, 0, 2ae^{45}+ae^{23}, 2ae^{24}, -ae^{25})$ with $T = \sqrt{3}ae^{135}$ and $k(\mathfrak{p}) = -3a^2$.

(4) $b_{45} = b_{19} = a, b_{39} = -b_{48} = -b, b_9 = -b_{24} = -c, a^2+b^2+c^2 \neq 0$
$(-ce^{35}, ae^{35}, ce^{15}-ae^{25}+be^{45}, -be^{35}, -ce^{13}+ae^{23}+be^{34})$ with
$T = (ce^1 - ae^2 + be^4)e^{35}$ and $k(\mathfrak{p}) = -a^2-b^2-c^2$.

The following observations guarantee that three cases are algebraically distinct. The first Lie algebra is 2-step solvable with $\dim \mathfrak{l}^1 = 3$, has an Abelian $\mathfrak{p}$ and $\mathfrak{h} \cong \mathbb{R} \oplus \mathfrak{s}_2$. The third one is 3-step solvable with 4-dimensional commutator. Numbers (2), (4) are both essentially $\mathfrak{so}(3) \oplus \mathbb{R}^2$, and account for $\mathfrak{g}_1$ in theorem 4.3. ☐

**Lemma 5.6.** *If $dT \neq 0$ and $(\mathfrak{l},\mathfrak{h})$ is a symmetric pair then $\mathfrak{l}$ is one of*

I. $(e^{24}, 0, -e^{23}, e^{24}, e^{25}+e^{34})$

II. $(6e^{24}, 2e^{14}, e^{15}-\sqrt{3}e^{23}-\sqrt{3}e^{45}, -2e^{12}, -e^{13}+\sqrt{3}e^{25}+\sqrt{3}e^{34})$

III. $\big((2c+\sqrt{3}(b^2+1))e^{24}+ce^{35}, -2be^{24}, -\frac{1}{2}(b^2+1)e^{23}-be^{25}-b^2e^{45}, (1-b^2)e^{24}, be^{23}+\frac{1}{2}(b^2+1)e^{25}+e^{34}\big)$

IV. $\big(-2(b^2+c^2)e^{24}-(b^2+c^2+1)e^{35}, -2be^{24}, e^{15}-be^{25}+ce^{45}, 2ce^{24}, -e^{13}+be^{23}+ce^{34}\big)$

*up to isomorphisms, where $b, c \in \mathbb{R}$. The surface $L/H$ has zero curvature in cases I, II and negative in the others.*

*Proof.* If $dT \neq 0$ the following possibilities come out of $d^2 = 0$:

(1) $b_6 = 2\sqrt{3}b_{47} = 2\sqrt{3}a \neq 0$;
$(2\sqrt{3}ae^{24}, 0, -ae^{23}-2ae^{45}, -2ae^{24}, ae^{25})$ and

$$T = -2\sqrt{3}a(e^{124}+e^{135}), \quad dT = 12a^2e^{2345}.$$

(2) $b_{48} = 2b_{47}, b_6 = 2\sqrt{3}b_{47}, b_{47} = a \neq 0$
$(2\sqrt{3}ae^{24}, 0, -ae^{23}, 2ae^{24}, ae^{25} + 2ae^{34})$ and
$$T = -2\sqrt{3}a(e^{124} + e^{135}), \quad dT = 12a^2e^{2345}.$$
(3) $b_{24} = \frac{1}{2}b_{13}, b_{48} = \frac{\sqrt{3}}{2}b_{13}, b_6 = 3b_{13}, b_{47} = \frac{\sqrt{3}}{2}b_{13}, b_{13} = a \neq 0$
$(3ae^{24}, ae^{14}, \frac{1}{2}ae^{15} - \frac{\sqrt{3}}{2}ae^{23} - \frac{\sqrt{3}}{2}ae^{45}, -ae^{12}, -\frac{1}{2}ae^{13} + \frac{\sqrt{3}}{2}ae^{25} + \frac{\sqrt{3}}{2}ae^{34})$ and
$$T = -3a(e^{124} + e^{135}), \quad dT = 9a^2e^{2345}.$$
(4) $b_9 = a, b_{45} = b, b_{48} = c \neq 0, b_6 = \frac{2ac+\sqrt{3}b^2+\sqrt{3}c^2}{c}, b_{47} = \frac{b^2+c^2}{2c}$
$(\frac{2ac+\sqrt{3}b^2+\sqrt{3}c^2}{c}e^{24} + ae^{35}, -2be^{24}, -\frac{b^2+c^2}{2c}e^{23} - be^{25} - \frac{b^2}{c}e^{45}, \frac{c^2-b^2}{c}e^{24}, be^{23} + \frac{b^2+c^2}{2c}e^{25} + ce^{34})$ with
$T = -\frac{2ac+\sqrt{3}b^2+\sqrt{3}c^2}{c}e^{124} - \frac{ac+\sqrt{3}b^2+\sqrt{3}c^2}{c}e^{135}, \quad dT = \frac{(2ac+\sqrt{3}b^2+\sqrt{3}c^2)^2}{c^2}e^{2345}$
and $k(\mathfrak{p}) = -a^2$. The algebra $\mathfrak{l}$ is 3-step solvable. For $b \neq 0$ the dimension of $\mathfrak{l}^1$ is four. Further asking $a \neq 0$ recovers [5, Example 6.3.2, $\delta = 1$]. Taking $b = 0$ gives a 3-dimensional commutator instead.
(5) $b_{24} = a \neq 0, b_{45} = b, b_{48} = c, b^2 + c^2 \neq 0, b_6 = -2\frac{b^2+c^2}{a}, b_9 = -\frac{b^2+c^2+a^2}{a},$
$(-2\frac{b^2+c^2}{a}e^{24} - \frac{b^2+c^2+a^2}{a}e^{35}, -2be^{24}, ae^{15} - be^{25} + ce^{45}, 2ce^{24}, -ae^{13} + be^{23} + ce^{34})$
$$T = 2\frac{b^2+c^2}{a}e^{124} + \frac{b^2+c^2+a^2}{a}e^{135}, \quad dT = 4\frac{(b^2+c^2)(a^2+b^2+c^2)}{a^2}e^{2345}.$$
This has 4-dimensional commutator such that $\mathfrak{l}^2 \cong \mathfrak{so}(3)$, and curvature $k(\mathfrak{p}) = -\big(\frac{b^2+c^2}{a} + 1\big)$. It corresponds to [5, Example 6.3.2] with $\delta = 0$.

The first two Lie algebras are 3-step solvable with 3-dimensional commutator and isomorphic. The third one is perfect, i.e. $\mathfrak{l}^1 = \mathfrak{l}$. For the Lie algebras (1)–(3), and for the family (4) with $a = 0$, $\mathfrak{p}$ is Abelian. The algebra $\mathfrak{h}$ is always $\mathfrak{s}_3$, except for (3) where $\mathfrak{h} \cong \mathfrak{so}(3)$. The first three instances fibre onto a Euclidean surface $L/H$, for the remaining ones $k(\mathfrak{p})$ is negative. Suitable normalisation of (3) and (5) give precisely $\mathfrak{l}_2, \mathfrak{l}_3$ of Theorem 4.3. □

The torsion has type $\Lambda^2_3$ for (4) with $-\sqrt{3}ac = b^2 + c^2$, and (5) with $a^2 = 3(b^2 + c^2)$. Up to $SO(3)$ equivalence these appear, though in disguise, in [5, Theorem 6.7].

Let $\widetilde{Ric}, Ric$ indicate the Ricci tensors of the characteristic and Levi-Civita connections. Although in general there is no reason for the former to be symmetric, this is what happens at present

**Proposition 5.7.** *If $(\mathfrak{l}, \mathfrak{h})$ is a symmetric pair, the torsion is always coclosed.*

*Proof.* One checks without effort that all Lie algebras in this section have a symmetric Ricci curvature $\widetilde{Ric}$. The general formula for a metric connection with skew torsion
$$\widetilde{Ric}(X,Y) = Ric(X,Y) + \tfrac{1}{2}{*d*}T(X,Y) + \tfrac{1}{4}\textstyle\sum_{i,j}^{1,5} g(T(X,e_i), T(Y,e_j))$$
allows to conclude. □

The same reasoning holds in the non-symmetric case. Since the characteristic connection will be identically zero there, every $SO(3)$-curvature tensor vanishes (theorem 7.1), turning $*T$ into a closed form.

Lemmas 5.6, 5.5 can be modified and proved differently, for it is actually possible to spot the closure of $dT$ a priori. A lengthy computation shows that $\mathfrak{S}_{x,y,z}(\tilde{\nabla}_x T)(y,z,w) = (\tilde{\nabla}_w T)(x,y,z)$ everywhere, so from (4.6) one has $dT = \sum (e_i \lrcorner T)^2$. Now $T$ can be either decomposable as $h \wedge e^{ij}$, for some $h \in \mathfrak{h}$. The contraction with any basis element is itself decomposable and simple, so $\sigma_T = 0$ eventually. Therefore the torsion must be strong by corollary 4.4. Alternatively, $T$ is proportional to $e^1 \wedge (e^{24} + e^{35})$, yielding a non-zero differential.

## 6. Parallel forms

6.1. **Parallel torsion.** Whenever in presence of a characteristic connection, it is relevant to consider whether it annihilates the torsion 3-form. Evidence of this very restrictive condition can be found in [7, 4]. In the former Cleyton and Swann prove that parallel torsion implies $d{*}T = 0$, which seems to pervade our classification. Precisely

**Proposition 6.1.** *Let $(\mathfrak{l}, \mathfrak{h})$ be a symmetric pair. Then $\mathfrak{l}$ admits (non-zero) parallel characteristic torsion iff it is isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$.*

*Proof.* As for lemma 5.5, when $T$ is closed (and coclosed) the Lie algebras (2) and (4) fulfill $\tilde{\nabla} T = 0$. An obvious coordinate change proves them both Abelian extensions of $\mathfrak{so}(3)$. If $dT \neq 0$ instead (see lemma 5.6), the torsion is never parallel. $\square$

6.2. **Reduced holonomy.** The holonomy of the characteristic connection will reduce to a subgroup only in presence of a parallel vector, and because of lack of space inside $SO(3)$ there is one non-trivial case, that of the circle

$$Hol(\tilde{\nabla}) = U(1) \iff \exists \xi \in \mathfrak{l} : \tilde{\nabla}\xi = 0.$$

The next result concerns groups $L$ other than the 5-torus:

**Proposition 6.2.** *Let $(\mathfrak{l}, \mathfrak{h})$ be a symmetric pair whose Lie group $L$ has characteristic holonomy $U(1) \subset SO(3)$ by way of a $\tilde{\nabla}$-parallel vector field $\xi$. Then $\mathfrak{l}$ is isomorphic to*

*1)* $\mathfrak{so}(3) \oplus \mathbb{R}^2$

*2)* $(0, e^{14}, e^{15}, e^{12}, e^{13})$,

*3)* $\left(-b_{48}(e^{24} + e^{13}) + b_9 b_{24} e^{35}, 0, e^{15}, e^{24}, e^{31}\right)$ *with* $b_{24} = -2(b_{45}^2 + b_{48}^2)/b_6$

*4)* $(0, 0, 0, 0, e^{15} + e^{23})$

*5)* $\mathfrak{s}_3 \oplus \mathbb{R}^2$

*6)* $(0, 0, e^{15}, e^{53}, e^{34})$.

*All others have full holonomy.*

*Proof.* First of all a parallel-vector-to-be $\xi$ must belong to $\mathfrak{h}$: the coefficients' relations imply that components in $e_3, e_5$ cannot appear. Disregarding situations where the connection itself is zero, imposing $\tilde{\nabla}\xi = 0$ yields either that

· $\mathfrak{p}$-derivatives are zero (precisely when $\xi = e_1$), or

· $\tilde{\nabla}_{\mathfrak{h}} \equiv 0$ (i.e. $\xi$ has also $e_2$-,$e_4$-components).

a) $e_1$ is parallel for three types of algebras:

$(0, 0, b_{24}e^{15} + b_{48}e^{45}, -b_{48}e^{35}, -b_{24}e^{13} + b_{48}e^{34})$

$\big(b_9e^{35}, (b_{19} - b_{45})e^{24} + b_{19}e^{35}, b_{24}e^{15} - b_{45}e^{25} + b_{48}e^{45}, -b_{48}e^{35}, -b_{24}e^{13} + b_{45}e^{23} + b_{48}e^{34}\big)$

$(b_6e^{24}, -b_6e^{14}, 0, b_6e^{12}, 0)$

$(0, b_{13}e^{14}, b_{24}e^{15}, -b_{13}e^{12}, -b_{24}e^{13}),$

$(b_6e^{24} + b_9e^{35}, -2b_{45}e^{24}, b_{24}e^{15} - b_{45}e^{25} + b_{48}e^{45}, 2b_{48}e^{24}, b_{24}e^{13} + b_{45}e^{23} + b_{48}e^{34})$

with $b_{24} = -2(b_{45}^2 + b_{48}^2)/b_6$. The first three are isomorphic.

b) When the line $\xi$ intersects the plane $\langle e_2, e_4\rangle$, the candidates for covariantly constant fields have the form

$$xe_1 + ye_2 + te_4,$$

subjected in particular to $t^2 + y^2 = 3x^2 \neq 0$ (we omit other relations imposed on (5.1)). Apart from the Abelian algebra $\mathfrak{a}_5$, here are the solutions:

$(b_9e^{35}, 0, b_9e^{51}, \frac{\sqrt{3}}{3}b_9e^{35}, b_9e^{13} - \frac{\sqrt{3}}{3}b_9e^{34}),$ $(b_6e^{24}, 0, b_6e^{41}, 0, b_6e^{12}),$

$\big(0, 0, b_{24}(e^{15} + \sqrt{3}e^{45}), 0, 0\big),$ $\big(0, 0, 0, 0, b_{45}(\sqrt{3}e^{15} + e^{23} + e^{45})\big),$

$(0, b_{19}e^{35}, b_{19}e^{52} + \sqrt{3}b_{39}e^{45}, b_{39}e^{35}, b_{19}e^{23} + b_{39}e^{34}).$

Recognising the isomorphisms is easy now. □

## 7. $(\mathfrak{l}, \mathfrak{h})$ non-symmetric pair

All examples in this section will have $\tilde\nabla = 0$. They have in particular flat characteristic connection, so their structure coincides with [5, eqns (6.10)]. We will also determine which ones have torsion of type $\Lambda^2_7$.

Remember that the $SO(3)$-connection $\tilde\nabla$ satisfies (4.2) if and only if

$$b_{43} = b_{10}, b_{23} = -b_8, b_{15} = b_{17} = b_{29} = b_{40} = b_{49} = 0.$$

One can assume at least one of $b_7, b_8, b_{10}, b_{12}, b_{35}, b_{37}$ to be non-zero to prevent $\mathfrak{p}$ from being $ad(\mathfrak{h})$-invariant and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{l}$. The structure equations then are

$$\begin{cases} de^1 &= -b_{12}e^{23} + b_6e^{24} + b_7e^{25} + b_8e^{34} + b_9e^{35} + b_{10}e^{45}, \\ de^2 &= b_{12}e^{13} + b_{13}e^{14} - b_7e^{15} + 2(b_{50} - b_{45} + b_{19})e^{24} + b_{35}e^{34} + b_{19}e^{35} - b_{37}e^{45}, \\ de^3 &= -b_{12}e^{12} + \sqrt{3}b_{28}e^{13} - b_8e^{14} + b_{24}e^{15} + \frac{\sqrt{3}}{3}(b_9 + b_{24} - \frac{1}{2}b_{13} - \frac{1}{2}b_6)e^{23} - b_{35}e^{24} + \\ & \quad (b_{50} + b_{28} - b_{45})e^{25} + b_{28}e^{34} + (\frac{\sqrt{3}}{3}b_9 + \frac{\sqrt{3}}{3}b_{24} - \frac{\sqrt{3}}{6}b_{13} - \frac{\sqrt{3}}{6}b_6 + b_{48} - b_{47})e^{45}, \\ de^4 &= -b_{13}e^{12} + b_8e^{13} - b_{10}e^{15} + b_{35}e^{23} + 2(b_{39} + b_{48} - b_{47})e^{24} + b_{37}e^{25} + b_{39}e^{35}, \\ de^5 &= b_7e^{12} + (b_9 + \sqrt{3}b_{47} - \frac{1}{2}b_{13} - \frac{1}{2}b_6)e^{13} + b_{10}e^{14} + \sqrt{3}b_{50}e^{15} + b_{45}e^{23} - b_{37}e^{24} + \\ & \quad b_{47}e^{25} + b_{48}e^{34} + b_{50}e^{45}, \end{cases}$$

and the torsion reads

$$\begin{aligned} T \;=\; & b_{12}e^{123} - b_6e^{124} - b_7e^{125} - b_8e^{134} + (b_{24} - \tfrac{1}{2}b_{13} - \tfrac{1}{2}b_6 - \sqrt{3}b_{47})e^{135} - \\ & b_{10}e^{145} - b_{35}e^{234} + (b_{28} - b_{50} - b_{19}e^{235} + b_{37}e^{245} + \\ & \tfrac{\sqrt{3}}{3}(\tfrac{1}{2}b_{13} + \tfrac{1}{2}b_6 - b_9 - b_{24} + b_{39} - b_{47})e^{345}. \end{aligned} \tag{7.1}$$

**Theorem 7.1.** *If* $(\mathfrak{l}, \mathfrak{h})$ *is non-symmetric, then* $\tilde{\nabla} = 0$ *and* $T$ *is harmonic*

$$dT = 0, \qquad d{*}T = 0.$$

*In addition,* $\mathfrak{l}$ *has* 3*-dimensional commutator* $\mathfrak{l}^1 = \mathfrak{l}^2$*.*

*Proof.* By imposing $d^2 = 0$ one gets the extra conditions:

$$b_{50} = b_{47} = b_{28} = 0, b_{48} = -b_{39}, b_{13} = -b_6, b_{24} = -b_9, b_{45} = b_{19},$$

and then $\tilde{\nabla} = 0$ by (3.3). Thus the above equations reduce to (6.10) of [5] with

$$t_1 = -b_{12}, t_2 = b_6, t_3 = b_7, t_4 = b_8, t_5 = b_9, t_6 = b_{10}, t_7 = b_{35}, t_8 = b_{19}, t_9 = 0, t_{10} = -b_{39}$$

subject to the constraints

$$\begin{cases} -b_{10}b_{19} + b_7 b_{39} - b_9 b_{37} = 0, \\ b_7 b_{35} - b_6 b_{19} + b_{12} b_{37} = 0, \\ -b_{12} b_{39} + b_8 b_{19} - b_9 b_{35} = 0, \\ -b_6 b_{39} + b_{10} b_{35} + b_8 b_{37} = 0, \\ -b_7 b_8 + b_{12} b_{10} + b_6 b_9 = 0. \end{cases}$$

This system implies $dT = d(*T) = 0$. This can be alternatively seen by computing the form $\sigma_T$ of (4.7), as done previously.

A standard computation yields

(1) $b_6 = -\frac{(-b_7b_8+b_{12}b_{10})}{b_9}, b_{37} = \frac{(b_7b_{39}-b_{19}b_{10})}{b_9}, b_{35} = -\frac{(-b_{19}b_8+b_{12}b_{39})}{b_9}, b_9 \neq 0,$
(2) $b_9 = 0, b_{35} = -\frac{(-b_6b_{19}+b_{12}b_{37})}{b_7}, b_8 = \frac{b_{12}b_{10}}{b_7}, b_{39} = \frac{b_{19}b_{10}}{b_7}, b_7 \neq 0,$
(3) $b_7 = 0, b_9 = 0, b_{12} = 0, b_{19} = 0, b_{35} = \frac{(-b_8b_{37}+b_6b_{39})}{b_{10}}, b_{10} \neq 0,$
(4) $b_7 = 0, b_{39} = \frac{b_{12}b_{37}}{b_6}, b_9 = b_{10} = 0, b_{39} = \frac{b_8b_{37}}{b_6}, b_6 \neq 0,$
(5) $b_{19} = \frac{b_{12}b_{39}}{b_8}, b_6 = b_7 = b_9 = b_{10} = b_{37} = 0, b_8 \neq 0,$
(6) $b_6 = b_7 = b_8 = b_9 = b_{10} = b_{12} = 0,$
(7) $b_6 = b_7 = b_8 = b_9 = b_{10} = b_{39} = 0,$

whose corresponding Lie algebras have all 3-dimensional commutator $\mathfrak{l}^1 = [\mathfrak{l}^1, \mathfrak{l}^1]$. Tedious details have been left out. $\square$

Since $\tilde{\nabla}$ is flat, [5, proposition 6.6] has $\mathfrak{l}$ isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$ provided that $b_{39} \neq 0$ in (7.1). We can generalise this result regardless of coefficients

**Theorem 7.2.** $\tilde{\nabla} = 0$ *on* $\mathfrak{l}$ *if and only if* $\mathfrak{l}$ *is isomorphic to* $\mathfrak{so}(3) \oplus \mathbb{R}^2$*.*

*Proof.* The connection is zero if and only if $g([X, Y], Z)$ is a totally skew. In other words if

$$g(\mathrm{ad}_X Y, Z) = -g(Y, \mathrm{ad}_X Z),$$

for any $X, Y, Z \in \mathfrak{l}$. Then $\mathfrak{l}$ is compact and Weyl's theorem says that $\mathfrak{l} = \mathfrak{l}^1 \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{l}$. By theorem 7.1 the commutator $\mathfrak{l}^1$ is a 3-dimensional centerless compact Lie algebra, whence semisimple. Thus it isomorphic to $\mathfrak{so}(3)$. $\square$

*Remarks 1.* The characteristic connection being zero is reminiscent of compact semisimple Lie groups of even dimension equipped with the complex structure of Samelson and metric equal to the negified Killing form. They have in fact zero Bismut connection and the corresponding torsion is always harmonic, exactly as in 7.1, see [17].

*2.* Note incidentally that if $\tilde{\nabla} \equiv 0$ then $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{p}$ defines a naturally reductive space, for both the characteristic curvature and the torsion are obviously parallel [18].

As equations (7.1) translate now in to

$$\sqrt{3}b_{35} = b_{10} - b_{12}, \quad b_8 + b_7 = \sqrt{3}b_{37}, \quad b_6 = -2b_9,$$

**Corollary 7.3.** *If* $(\mathfrak{l}, \mathfrak{h})$ *is non-symmetric, the torsion* $T$ *is of pure type* $\Lambda^2_7$ *when:*

(1) $b_6 = -2b_9, b_{12} = -\sqrt{3}b_{35} + b_{10}, b_{39} = \frac{\sqrt{3}}{6b_9}(2b_9^2 - b_{10}^2 - b_8^2), b_{37} = -\frac{\sqrt{3}}{3b_8}(-b_8^2 + 2b_9^2 - 3b_{10}^2 + \sqrt{3}b_{10}b_{35}), b_{19} = \frac{8\sqrt{3}}{3}(-b_{10}b_8^2 + 2b_{10}b_9^2 - b_{10}^3 + \sqrt{3}b_{10}^2 b_{35} + \sqrt{3}b_8^2 b_{35}), b_7 = -\frac{1}{b_8}(2b_9^2 - b_{10}^2 + \sqrt{3}b_{10}b_{35})$
(2) $b_6 = -2b_9, b_8 = 0, b_{19} = -\frac{b_{10}b_{37}}{2b_9}, b_{39} = \frac{\sqrt{3}}{6b_9}(2b_9^2 - b_{10}^2), b_{35} = -\frac{\sqrt{3}}{3b_{10}}(2b_9^2 - b_{10}^2), b_{12} = \frac{2b_9^2}{b_{10}}, b_7 = \sqrt{3}b_{37}$
(3) $b_8 = b_{35} = b_{39} = 0, b_{10}^2 = 2b_9^2, b_6 = -2b_9, b_{12} = b_{10}, b_7 = \sqrt{3}b_{37}, b_{37}^2 = 2b_{19}^2$
(4) $b_6 = b_8 = b_9 = b_{10} = b_{39} = 0, b_{12} = -\sqrt{3}b_{35}, b_7 = \sqrt{3}b_{37}$
(5) $b_6 = b_7 = b_8 = b_9 = b_{10} = b_{12} = b_{35} = b_{37} = 0.$ □

## 8. Example of pure type $\Lambda^2_7$ with non-closed torsion

The majority of Lie algebras found are strongly *NI*, in the sense that $T$ is closed. It is tempting to think this feature could be proven in general, but this is not the case. The disproving example is constructed using a fibration similar to (4.3) and will be required to have type $\Lambda^2_7$.

We consider Lie algebras $\mathfrak{l}$ whose Levi-Civita connection satisfies

$$\begin{array}{lll} \nabla_X \mathfrak{h} \subseteq \mathfrak{p}, & \nabla_X \mathfrak{p} \subseteq \mathfrak{h}, & \forall X \in \mathfrak{h} \\ \nabla_Y \mathfrak{h} \subseteq \mathfrak{h}, & \nabla_Y \mathfrak{p} \subseteq \mathfrak{p}, & \forall Y \in \mathfrak{p}. \end{array} \tag{8.1}$$

These are in some sense 'dual' to those of proposition 5.1. The constraints (8.1) are equivalent to $(\mathfrak{l}, \mathfrak{p})$ being a symmetric pair this time, i. e.

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{p}.$$

The 1-connected Lie subgroup $P \subset L$ with Lie algebra $\mathfrak{p}$ is the typical fibre of the Riemannian submersion

$$L \longrightarrow L/P$$

that renders $L/P$ a 3-dimensional locally symmetric space. With the aid of O'Neill formulas the Ricci tensor can be expressed via

$$Ric(X, X) = -([X, [X, e_1]_{\mathfrak{p}}]_{\mathfrak{h}}, e_1) - ([X, [X, e_2]_{\mathfrak{p}}]_{\mathfrak{h}}, e_2) - ([X, [X, e_4]_{\mathfrak{p}}]_{\mathfrak{h}}, e_4),$$

for any $X \in \mathfrak{h}$, and yields the scalar curvature in terms of the $b_i$'s.

The Lie algebra

$$\mathfrak{l} = (-\tfrac{3}{4}e^{15} + \tfrac{3\sqrt{3}}{4}e^{23} - \tfrac{\sqrt{3}}{4}e^{45}, e^{25}, \sqrt{3}e^{12} - e^{24} - e^{35}, -\tfrac{5}{4}\sqrt{3}e^{15} - \tfrac{9}{4}e^{23} - \tfrac{5}{4}e^{45}, 0)$$

is a particular solution of the system (4.5) coupled with the Jacobi identity. It has highest step length (four) and commutator $\mathfrak{l}^1 = \mathrm{span}\{e_1, \dots, e_4\}$. Its torsion

$$T = \tfrac{\sqrt{3}}{4}e^{123} - \sqrt{3}e^{145} - \tfrac{3}{4}e^{234}$$

is not strong

$$dT = -\tfrac{3}{2}(e^{2345} + \sqrt{3}e^{1235}), \quad d{*}T = 0.$$

The Ricci curvature

$$Ric(L) = \begin{pmatrix} -\frac{141}{32} & 0 & 0 & -\frac{99}{32}\sqrt{3} & 0 \\ 0 & -\frac{27}{8} & 0 & 0 & 0 \\ 0 & 0 & -\frac{27}{8} & 0 & 0 \\ -\frac{99}{32}\sqrt{3} & 0 & 0 & -\frac{99}{32}\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & -\frac{15}{2} \end{pmatrix}$$

is negative, and $L/P$ is scalar-flat.

## 9. Higher dimensional geometry

Although $SO(3)$ geometry is quite intriguing on its own, it is the links with other $G$-structures that make it really valuable. From example 3.6 in fact, one can build 6-manifolds with holonomy $SU(3)$, a feature shared by probably many other instances of the same kin.

A direct jump to dimension 7 is possible, and twofold alluring: first because there exist non-integrable CR-structures arising from 5-manifolds of pure type $\Lambda^2_7$. Secondly, dimension seven is also inhabited by $G_2$ metrics, and it is all the more natural to consider 2-sphere bundles over $L$ related to a twistor theory of sorts [5].

We wish to concentrate on dimension eight now though, motivated by [14]. There the author investigates $G$-structures related to a series of simple Lie groups $SO(3)$, $SU(3), \ldots$ acting in dimensions $5, 8, \ldots$, describes the corresponding *NI* conditions and discusses the existence of a characteristic connection. We thus consider on the 8-dimensional product $L \times K$, with $K = \mathbb{R}^3$ or $SO(3)$, the following left-invariant metric

$$\tilde{g} = g + (e^6)^2 + (e^7)^2 + (e^8)^2,$$

where $g$ is the left-invariant metric on $L$ of page 3 and $\{e_6, e_7, e_8\}$ is a basis of the Lie algebra $\mathfrak{k}$ of $K$. When $K = SO(3)$ it is the standard basis $[e_6, e_7] = e_8, [e_7, e_8] = e_6, [e_8, e_6] = e_7$. The vector space $\mathbb{R}^8$ is identified with the set of traceless skew-Hermitian $3 \times 3$ matrices by way of

$$\tilde{X} = (x_1, \ldots, x_8) \leftrightarrow \begin{pmatrix} x_1 - \sqrt{3}x_4 & \sqrt{3}(x_2 + ix_8) & \sqrt{3}(x_3 + ix_7) \\ \sqrt{3}(x_2 - ix_8) & x_1 + \sqrt{3}x_4 & \sqrt{3}(x_5 + ix_6) \\ \sqrt{3}(x_3 - ix_7) & \sqrt{3}(x_5 - ix_6) & -2x_1 \end{pmatrix}.$$

The irreducible representation of $SU(3)$ on $\mathbb{R}^8$ is given – as for $SO(3)$ – by $\tilde{\rho}(h)\tilde{X} = h\tilde{X}h^{-1}, h \in SU(3)$. An $SU(3)$ structure on $(L \times K, \tilde{g})$ is defined [14] by an element $\tilde{\mathbb{T}} \in \otimes^3\mathbb{R}^8$ satisfying similar relations to those of $\mathbb{T}$

$$\begin{aligned} \tilde{\mathbb{T}}(\tilde{X}, \tilde{X}, \tilde{X}) &= \mathbb{T}(X, X, X) + \tfrac{3}{2}x_1(x_6^2 + x_7^2 - 2x_8^2) - \tfrac{3\sqrt{3}}{2}x_4(-x_7^2 + x_6^2) \\ &\quad + 3\sqrt{3}(x_3x_6x_8 - x_5x_6x_7 + x_2x_7x_6), \\ &= \tfrac{1}{2}\det \tilde{X} \end{aligned}$$

with $X = (x_1, \ldots, x_5)$. Setting $x_6, x_7, x_8$ to zero induces the same structure on $\mathbb{R}^5 \subset \mathbb{R}^8$ as (2.1), because the position of $\sqrt{3}$ is only cosmetic. Yet in stark contrast to dimension five [5], the reduction is equivalently determined by the differential 3-form

$$\psi = E_1 \wedge e^6 + E_2 \wedge e^7 + E_3 \wedge e^8 + e^{678},$$

where the $E_j$'s come from (2.7). The isotropy of $\psi$ is $SU(3)/\mathbb{Z}_3$ embedded in $GL(8,\mathbb{R})$. We refer the reader to [12] where the action of $SU(3)$ on $\mathbb{R}^8$ was first examined in detail. All details in the same flavour can be found in [19]. As in the lower dimension, a nearly integrable $SU(3)$ structure is defined in terms of a symmetric Killing tensor $\tilde{\mathbb{T}}$. This only implies the existence of a connection with totally skew torsion, but is no longer equivalent to it. We shall see under which circumstances one can get hold of a nearly integrable $SU(3)$ structure on $L\times K$, and that the induced characteristic connection is the zero connection, making $\psi$ obviously parallel.

The explicit components of $\tilde{\mathbb{T}}=\sum_{i,j,k=1}^{8}\tilde{t}_{ijk}\,dx_i\otimes dx_j\otimes dx_k$ are:

$$\begin{gathered}\tilde{t}_{ijk}=t_{ijk},\quad i,j,k\leqslant 5;\\ \tilde{t}_{166}=-\tfrac{1}{2}=\tilde{t}_{177},\quad \tilde{t}_{188}=1,\quad \tilde{t}_{466}=-\tilde{t}_{477}=\tilde{t}_{267}=-\tilde{t}_{368}=\tilde{t}_{578}=\tfrac{\sqrt{3}}{2}.\end{gathered}\tag{9.1}$$

Let $\nabla$ indicate the Levi-Civita connection on $\mathfrak{l}+\mathfrak{k}$ and also its restriction to $\mathfrak{l}$ and $\mathfrak{k}$, so

$$\nabla_{e_6}e_7=\tfrac{1}{2}e_8=-\nabla_{e_7}e_6,\quad \nabla_{e_6}e_8=-\tfrac{1}{2}e_7=-\nabla_{e_8}e_6,\quad \nabla_{e_7}e_8=\tfrac{1}{2}e_6=-\nabla_{e_8}e_7$$

for $K=SO(3)$.

**Lemma 9.1.** *Let $(L^5,g,\mathbb{T})$ be NI. Then the $SU(3)$ structure $(L\times K,\tilde{g},\tilde{\mathbb{T}})$, with $K=\mathbb{R}^3$ or $SO(3)$ is NI if and only if*

$$\nabla_X X=0\quad \textit{for all } X\in\mathfrak{l}.$$

*Proof.* Set $\tilde{X}=X+Y\in\mathfrak{l}+\mathfrak{k}$. By construction $\nabla_X Y=0$, and both when $\mathfrak{k}=\mathbb{R}^3$ and $\mathfrak{k}=\mathfrak{so}(3)$, $\nabla_Y Y$ vanishes as well. Thus

$$\begin{aligned}0&=\tilde{\mathbb{T}}(\nabla_{\tilde{X}}\tilde{X},\tilde{X},\tilde{X})=\tilde{\mathbb{T}}(\nabla_X X,\tilde{X},\tilde{X})\\&=\mathbb{T}(\nabla_X X,X,X)+2\tilde{\mathbb{T}}(\nabla_X X,X,Y)+\tilde{\mathbb{T}}(\nabla_X X,Y,Y).\end{aligned}$$

The first term (all $X$'s) vanishes by near integrability of $L$, and the second too because the components (9.1) yield $\tilde{\mathbb{T}}(\mathfrak{l},\mathfrak{l},\mathfrak{k})=0$. By linearity $\tilde{\mathbb{T}}(\nabla_X X,e_i,e_j)=0,\ i,j=6,7,8$. Using (9.1) once more gives $g(\nabla_X X,\mathfrak{l})=0$. The argument also works backwards. □

The relation of the lemma says that $\nabla_{e_i}e_j+\nabla_{e_j}e_i=0$ for all $i,j=1,\dots,5$, which implies (3.4). The first string of which characterises (4.2), so

**Corollary 9.2.** *The geodesic equation $\nabla_X X=0$ induces the split behaviour of $\tilde{\nabla}$ on $\mathfrak{l}$ described by (4.2) and forces $\tilde{\nabla}$ to be zero.* □

In this case, the skew-symmetry of $\nabla$ can be simply detected by considering the relation $2g(\nabla_{e_i}e_j+\nabla_{e_j}e_i,e_k)=T(e_i,e_j,e_k)+T(e_j,e_i,e_k)$, $\forall i,j,k$.
That said, generating nearly integrable $SU(3)$ structures on the 8-manifold $(L\times K,\tilde{g},\tilde{\mathbb{T}})$ is possible only by means of one candidate

**Proposition 9.3.** *Up to isomorphisms, the unique symmetric pair giving rise to NI products $L\times\mathbb{R}^3, L\times SO(3)$ is $\big(\mathfrak{l}=\mathfrak{so}(3)\oplus\mathbb{R}^2,\mathfrak{h}=\mathfrak{so}(3)\big)$.*

*Proof.* In the symmetric case $\nabla_X X = 0$ reduces the structure equations of $L$ to

$$\begin{cases} de^1 = b_6 e^{24} + b_9 e^{35} \\ de^2 = -b_6 e^{14} + b_{19} e^{35} \\ de^3 = e^5 \wedge (b_9 e^1 + b_{19} e^2 + b_{39} e^4) \\ de^4 = b_6 e^{12} + b_{39} e^{35} \\ de^5 = (b_9 e^1 + b_{19} e^2 + b_{39} e^4) \wedge e^3. \end{cases}$$

From the Jacobi identity either $b_6 = 0$ or $b_9 = b_{19} = b_{39}$. Both structures are isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}^2$ and crop up as II, IV in lemma 5.5. ☐

**Proposition 9.4.** *Non-symmetric pairs with $b_{37} = 0$ always induce a nearly integrable $SU(3)$ structure on the product $L \times K$.*

*Proof.* All algebras of section 7 satisfy, by way of $d^2 = 0$, the extra requirements $b_{50} = b_{47} = b_{28} = 0, b_{48} = -b_{39}, b_{13} = -b_6, b_{24} = -b_9, b_{45} = b_{19}$. ☐

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(A.Fino) Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy

*E-mail address*: `fino@dm.unito.it`

(S.Chiossi) Institut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

*E-mail address*: `sgc@math.hu-berlin.de`